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# On Lipschitz Conditions, Strong Unicity and a Theorem of A. K. Cline

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## 1. INTRODUCTION

Let $C(X)$ denote the set of continuous real valued functions on a locally compact Hausdorff space $X$. In a recent paper [3] A. K. Cline studied Lipschitz conditions on the best Chebyshev approximation operator. There is an oversight in the proof in [3] of the following interesting result:

CLINE'S THEOREM. *Let $X$ be a finite point set and let $M$ be a Haar subspace of $C(X)$. Then there is a constant $K$ (depending only on $X$ and $M$) such that for any $f$ and $g$ in $C(X)$, the best approximations (from $M$) to $f$ and $g$, $P(f)$ and $P(g)$, respectively, satisfy*

$$\| P(f) - P(g)\| \leqslant K \|f - g\|. \tag{1}$$

The proof given in [3] depends upon the assertion that the strong unicity constant $\gamma(f)$ is continuous. This is false even when $X$ is finite, as we will show later on; however, there are some continuity—like properties of $\gamma(f)$. In this note, we first give a correct proof of Cline's Theorem, and then discuss the continuity properties of $\gamma(f)$.

## 2. PRELIMINARIES

Let $M$ denote a finite dimensional Haar subspace of $C(X)$. The cases of most interest are when $X$ is finite, and when $X = [0, 1]$. Let $\|f\|$ denote the Chebyshev (uniform) norm of $f$ on $X$ and let $P(f)$ denote the best approximate to $f$ from $M$.

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DEFINITION. A function $\pi \in M$ is said to be a strongly unique best approximate to a given function $f$ in $C(X)$ if there exists a real number $r > 0$ such that

$$\|f - m\| \geqslant \|f - \pi\| + r\|m - \pi\|, \qquad \text{for all} \quad m \in M.$$

Let $\gamma(f)$ be the largest such $r$.

D. J. Newman and H. S. Shapiro [4] introduced the concept of strong uniqueness and proved that the best approximate to a function $f$ in $C(X)$ from a Haar subspace is strongly unique.

A result of Freud [2, p. 82] states that for $M$ Haar, the best approximation operator $P$ satisfies at each point a Lipschitz condition, i.e., if $f \in C(X)$, then there exists a constant $K$ such that for any $g \in C(X)$, Eq. (1) holds where $K = 2(\gamma(f))^{-1}$. The paper by Cline makes a study of Eq. (1).

Let $S(M) = \{m \in M : \|m\| = 1\}$. The strong Kolmogorov criterion [1] characterizing strongly unique best approximates states that

$$\gamma(f) = \inf_{m \in S(M)} \max_{x \in E(f)} [f(x) - P(f)(x)] \|f - P(f)\|^{-1} m(x), \tag{2}$$

where $E(F) = \{x \in X : |f(x) - P(f)(x)| = \|f - P(f)\|\}$.

## 3. RESULTS

We give a corrected proof of Cline's Theorem.

*Proof.* We show that

$$\gamma = \inf_{f \in C(X)} \gamma(f)$$

satisfies $\gamma > 0$. Therefore, for any $f, g \in C(X)$,

$$\|P(f) - P(g)\| \leqslant 2\gamma^{-1} \|f - g\|.$$

To demonstrate that $\gamma > 0$, we show that there are only finitely many values which $\gamma(f)$ can assume, none of which can be zero.

Let $g$ be in $C(X)$ and have the strongly unique best approximation $P(g)$. Then letting $g_1 = (g - P(g)) \|g - P(g)\|^{-1}$ we have $P(g_1) = 0$ and $\|g_1\| = 1$. Also by Proposition 1 in [1], $\gamma(g_1) = \gamma(g)$. Therefore without loss of generality we assume that $\|g\| = 1$ and $P(g) = 0$. Since $E(g)$ has at least one point, $E(g)$ is one of at most $\sum_{r=1}^{N} \binom{N}{r}$ sets, where $N$ is the number of points in $X$. If $E(g)$ has $\binom{N}{r}$ points $1 \leqslant r \leqslant N$, then $g = \pm 1$ at the points in $E(g)$ and $g$ was defined in one of at most $\exp\binom{N}{r} \log 2$ ways on $E(g)$. According to the strong Kolmogorov criterion, $\gamma(g)$ is determined by the values $g$ assumes on $E(g)$. Therefore $\gamma(g)$ can have at most finitely many values.

We now examine the continuity properties of $\gamma(f)$, both for $X = [0, 1]$ and for $X$ finite and find that in general, $\gamma(f)$ can be badly discontinuous. As observed in [2], we have $0 < \gamma(f) \leqslant 1$. Let $\langle 1, x\rangle$ denote the subspace of $C[0, 1]$ spanned by 1 and $x$. Now $\gamma(f)$ might be discontinuous on $M$ and continuous off $M$ (for a similar result see Theorem 3). The following result shows that in general $\gamma(f)$ need not be continuous off $M$.

THEOREM 1. *Let $X = [0, 1]$ and $M = \langle 1, x\rangle$. Then given $0 < \delta < 1$ and $\epsilon > 0$, there exist functions $f$ and $g$ in $C[0, 1]$ such that $f \notin M$ and $g \notin M$ and*

$$\|f - g\| < \epsilon \qquad \text{and} \qquad \gamma(f) - \gamma(g) > \delta.$$

*Proof.* Define $f_n(x)$ in $C[0, 1]$ for $n > 5$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0, 1/5, 1/2, 1 \\ -1 & \text{if } x = 1/n, 1/4, 1 - (1/n) \end{cases}$$

and $f_n$ is linear in between these points. Now

$$\|f_n\| = 1, \quad P(f_n) = 0 \quad \text{and} \quad E(f_n) = \{0, 1/n, 1/5, 1/4, 1/2, 1 - (1/n), 1\}.$$

Let $m \in M$ satisfy $\|m\| = 1$. Then $|m(x)| = 1$ at $x = 0$ or at $x = 1$. Checking each of the four possibilities separately by using (2) we see that $\gamma(f_n) \geqslant 1 - (2/n)$. For instance, if $m(0) = -1$, and $m(x) = cx - 1$, then $f_n(1/n)\, m(1/n) = |1 - (c/n)| \geqslant 1 - (2/n)$. Now define $g_n \in C[0, 1]$ by

$$g_n(x) = \begin{cases} f_n(x) & \text{if } x \notin [(1/4) - (1/40n), (1/4) + (1/8n)] \\ & \qquad \cup\, [1 - (1/n) - (n-2)/(4n^2), 1 - (1/n) + (1/2n^2)] \\ -1 + (1/n) & \qquad \text{otherwise.} \end{cases}$$

Then

$$\|g_n\| = 1, \qquad P(g_n) = 0 \qquad \text{and} \qquad E(g_n) = \{0, 1/n, 1/5, 1/2, 1\}.$$

If $m(x) = -x$, then

$$\max_{x \in E(g_n)} g_n(x)\, m(x) = 1/n.$$

Thus $\gamma(g_n) \leqslant 1/n$ and $\|f_n - g_n\| = 1/n$.

EXAMPLE 1. To see that $\gamma(f)$ need not be continuous when $X$ is finite, let $X = \{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{2}, \frac{5}{6}, 1\}$, let $f(x) = f_6(x)$ as defined above, and let $M = \langle 1, x\rangle$. As above for $n = 1, 2,\ldots$, let $g_n(x) = f(x)$ for all $x$ except $\frac{1}{4}$ and $\frac{3}{4}$ where $g_n(x) = -1 + (1/n)$. Then $\gamma(f) \geqslant \frac{2}{3}$ and $\gamma(g_n) \leqslant \frac{1}{6}$. Thus $\{g_n\}$ converges uniformly to $f$ but $\lim_{n\to\infty} \gamma(g_n) \leqslant \frac{1}{6}$.

Although $\gamma(f)$ is not in general continuous, we see in the next two results that $\gamma$ has some characteristics of continuity. At the Regional Conference on the Theory of Best Approximation and Functional Analysis held at Kent State University from June 11 to June 15, 1973, R. R. Phelps commented that $\gamma$ is an upper semicontinuous function. Here is a proof of that result. The proof does not depend on $M$ being a Haar set. Thus for any finite dimensional subspace $M$, $\gamma$ is upper semicontinuous on the subspace of $C(X)$, where it is defined.

THEOREM 2 (R. R. Phelps). *The strong unicity constant $\gamma(f)$ is an upper semicontinuous function.*

*Proof.* It must be shown that if a sequence $\{g_n\}$ converges uniformly to $f$, then

$$\limsup_{n\to\infty} \gamma(g_n) \leqslant \gamma(f).$$

Assume to the contrary that there is a sequence $\{g_n\}$ converging to $f$ and an $\epsilon > 0$ such that $\gamma(g_n) > \gamma(f) + \epsilon$ for all $n$. Now

$$\|g_n - m\| \geqslant \|g_n - P(g_n)\| + \gamma(g_n)\|P(g_n) - m\| \qquad \text{for all} \quad m \in M.$$

Fix $m \in M$. Then

$$\|g_n - m\| \geqslant \|g_n - P(g_n)\| + (\gamma(f) + \epsilon)\|P(g_n) - m\|$$

and letting $n \to \infty$ we find

$$\|f - m\| \geqslant \|f - P(f)\| + (\gamma(f) + \epsilon)\|P(f) - m\|$$

which holds now for any $m$ in $M$ and this contradicts the definition of $\gamma(f)$.

Observe that Theorem 1 depends on $E(f_n)$ and $E(g_n)$ not being "near." For a measure of nearness between any two subsets $A$ and $B$ of the metric space $(X, \rho)$ we use

$$d(A, B) = \sup_{y\in B} \inf_{x\in A} \rho(x, y).$$

Of course this is not a distance since $d(A, B)$ need not equal $d(B, A)$. But $d(A, B)$ does measure the "denseness" of $A$ in $B$. This measure of contiguity permits one to recover some aspects of continuity in the behavior of $\gamma(f)$ off $M$, but not in general on $M$.

THEOREM 3. *Let $M$ be a Haar subspace of $C(X)$ where $X$ is a compact metric space. Let $\{f_n\}$ be a sequence in $C(X)$ converging uniformly to $f$ where $f \notin M$. Assume that* $\lim_{n\to\infty} d(E(f_n), E(f)) = 0$. *Then*

$$\lim_{n\to\infty} \gamma(f_n) = \gamma(f).$$

*Proof.* For ease of writing we give the proof for $X = [0, 1]$ with the usual metric. Since $f \notin M$, we can assume without loss of generality that $f_n \notin M$ for all $n$. Also since $(f_n - P(f_n))\|f_n - P(f_n)\|^{-1}$ converges to $(f - P(f))\|f - P(f)\|^{-1}$ we may assume without loss of generality (by Proposition 1 of [1]) that $\|f_n\| = \|f\| = 1$ and $P(f_n) = P(f) = 0$ for each $n$. Let $\epsilon > 0$ be given. By Theorem 2, it suffices to show that there exists an $N$ such that $\gamma(f) \leqslant \gamma(f_n) + \epsilon$ for all $n > N$. Now $S(M)$ is uniformly equicontinuous on $X$ [4]. Let $\delta > 0$ be such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/5$ and also $|m(x) - m(y)| < \epsilon/5$ for all $m \in S(M)$. Let $N$ be such that if $n > N$, then $\|f_n - f\| < \epsilon/5$ and $d(E(f_n), E(f)) < \delta$. Fix $n > N$. Then there exists an $m_n \in S(M)$ such that

$$-\gamma(f_n) + \max_{x_n \in E(f_n)} f_n(x)\, m_n(x) \leqslant \epsilon/5.$$

Thus for a fixed $m_1(x) \in S(M)$

$$\begin{aligned} \gamma(f) - \gamma(f_n) &\leqslant \gamma(f) - \max_{x \in E(f)} f(x)\, m_1(x) - \gamma(f_n) + \max_{x_n \in E(f_n)} f_n(x)\, m_n(x) \\ &\quad + \max_{x \in E(f)} f(x)\, m_1(x) - \max_{x_n \in E(f_n)} f_n(x)\, m_n(x) \\ &\leqslant (\epsilon/5) + (\max_{x \in E(f)} f(x)\, m_1(x) - \max_{x_n \in E(f_n)} f_n(x)\, m_n(x)). \end{aligned}$$

Given $x' \in E(f)$, there exists an $x_n'$ in $E(f_n)$ such that $|x_n' - x'| < \delta$. Thus

$$\begin{aligned} &|f(x')\, m_1(x') - f_n(x_n')\, m_n(x_n')| \\ &\quad\leqslant |f(x')\, m_1(x') - f(x_n')\, m_1(x')| + |f(x_n')\, m_1(x') - f_n(x_n')\, m_1(x')| \\ &\qquad + |f_n(x_n')\, m_1(x') - f_n(x_n')\, m_n(x_n')| \\ &\quad\leqslant (2\epsilon/5) + \|f_n\|\, |m_1(x') - m_n(x_n')| \\ &\quad\leqslant 4\epsilon/5. \end{aligned}$$

Thus $\gamma(f) - \gamma(f_n) \leqslant \epsilon$ and we are done.

In Theorem 3, if $f \in M$ and if $f_n \in M$ for all $n$, then $\gamma(f_n) = \gamma(f) = 1$ for all $n$. But in general we see next that the conclusion does not follow in Theorem 3 if $f \in M$.

EXAMPLE 2. Let $M$ be the subspace $\langle 1, x \rangle$ of $C[0, 1]$. Define $f_n(x)$ in $C[0, 1]$ by

$$f_n(x) = \begin{cases} 1/n & \text{if } x = 0, 1/4n, 1/2n \\ -1/n & \text{if } x = 1/3n, 1/n, 1 \end{cases}$$

and $f_n$ is linear inbetween these points. Then $\|f_n\| = 1/n$ and $P(f_n) = 0$. Thus $\{f_n\}$ converges uniformly to zero and $\gamma(0) = 1$. But $\gamma(f_n) \leqslant (2n^2)^{-1}$. Indeed, let $m(x) = x$, then

$$\max_{x \in E(f_n)} f_n(x)\, m(x) = (2n^2)^{-1}.$$

However $\lim_{n \to \infty} d(E(f_n), E(f)) = 0$ since $[1/n, 1] \subseteq E(f_n)$.

## References


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